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Cohen–Macaulay Connectivity and Geometric Lattices

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INTRODUCTION

In this paper we introduce a new notion of connectivity for simplicial complexes and for partially ordered sets (posets), called *Cohen–Macaulay connectivity*. The concept of a Cohen–Macaulay (CM) poset was introduced by the author in [1], and the theory was developed in [3] and [7]. As remarked in [3, Section 8], the CM property is essentially a sophisticated connectivity property of a poset. It seems natural to consider higher connectivity in this setting. Moreover, there are two important classes of examples where this concept does arise naturally.

The first class we consider is the class of "circuit matroids" of graphs. By a method described in Section 1, we associate a CM complex $\Delta(\Gamma)$ to any graph Γ. A graph Γ is said to be *k-connected* if for any subset $T \subseteq E$ such that $|T| < k$, we have that $\Gamma \backslash T$ is a connected subgraph of Γ. Our definition of k-CM connectivity includes k-connectivity of graphs (for $k \geqslant 2$) as a special case. We discuss this fact and other basic properties of k-CM connectivity in Section 2.

Geometric lattices form another important class of CM posets. In fact, a geometric lattice L has an even stronger property: if $x \in L$, then the poset $L\backslash\{x\}$ is also CM. Thus a geometric lattice is "doubly Cohen–Macaulay" in a sense analogous to double connectivity of a graph. One of our main results (Section 3) is a simple criterion for a geometric lattice to be "k-connected" with respect to the CM property.

There now exist quite a variety of equivalent ways to define the concept of a CM complex. These criteria make use of combinatorial and ring-theoretic as well as topological concepts. In Section 4, we discuss what these various criteria mean for CM connectivity. Our main result is a ring-theoretic criterion for CM connectivity.

Yet another context within which CM connectivity appears is the theory of Coxeter complexes and Tits buildings as discussed by Björner [10]. For other uses of CM connectivity see [4, 5, 6]. Lemma 4.6 is of some interest in itself: closely related results were independently discovered by both Björner [9] and Stanley [19].

1. PRELIMINARIES

We make the overall assumption in this paper that all simplicial complexes and posets are finite. A simplicial complex Δ on vertex set V is any collection of subsets of V called simplices, which is closed under inclusion and contains $\varnothing$. Note that we do not require that $\{v\} \in \Delta$ for every $v \in V$.

A poset P defines a simplicial complex $\Delta(P)$, called the *order complex* of P, whose vertex set is P and whose simplices are the chains (totally ordered subsets) of P. Conversely, if Δ is a simplicial complex, then the non-empty simplices of Δ form a poset $P(\Delta)$. Note that we do *not* regard Δ itself as being a poset. It is essential to distinguish between simplicial complexes and posets in the context of this paper.

Let Δ be a simplicial complex and P a poset. The *rank* of a simplex $\sigma \in \Delta$ is its cardinality; the rank of Δ is $r(\Delta) = \max\{|\sigma| \mid \sigma \in \Delta\}$. To extend these concepts to posets we need some notation. We write $\hat{P}$ for the poset $P \cup \{\hat{0}, \hat{1}\}$ for which $\hat{0} < x < \hat{1}$ for

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every $x \in P$. The rank of P is the largest cardinality of the chains it contains, i.e. $r(P) = r(\Delta(P))$. The rank $r(x)$ of an element $x \in P$ is the rank of the half-open interval $(\hat{0}, x]$ in $\hat{P}$. We say that Δ is *pure* if every maximal simplex of Δ has rank $r(\Delta)$, and that P is *ranked* (or *graded*) if $\Delta(P)$ is pure.

Let Δ be a simplicial complex on vertex set V. For $T \subseteq V$, the *restriction* of Δ to T is the simplicial complex

$$\Delta|T = \{\sigma \in \Delta \mid \sigma \subseteq T\}.$$

We write $\Delta\backslash T$ for $\Delta|(V\backslash T)$. We say that Δ is *totally pure* if for every $T \subseteq V$, $\Delta|T$ is pure. A totally pure complex is also called a *matroid*.

Let $\Gamma = (W, E)$ be a graph on vertex set W with edge set E. We allow loops and multiple edges. The *circuit matroid* of Γ is the totally pure complex $\Delta(\Gamma)$ on E, whose simplices are the subsets of E not containing a circuit. A graph not containing a circuit is called a *forest*. A connected forest is called a *tree*. The circuit matroid of Γ necessarily loses some of the information contained in Γ. For example, $\Delta(\Gamma)$ alone will not determine whether Γ is connected. However, the higher connectivity of Γ is computable from $\Delta(\Gamma)$. We say that a connected graph Γ is *k-(edge) connected* if for any subset $T \subseteq E$ such that $|T| < k$, the subgraph of Γ defined by $E\backslash T$ is connected. It is easy to verify that Γ is k-connected if and only if $r(\Delta(\Gamma)\backslash T) = r(\Delta(\Gamma))$ for every $T \subseteq E$ such that $|T| < k$.

We assume the reader is familiar with the *reduced simplicial homology* of a simplicial complex. To avoid cumbersome notation we will always compute homology with coefficients in a field K, arbitrary but fixed throughout the paper. We write $\tilde{h}_m(\Delta)$ for the mth (reduced) Betti number of Δ; that is,

$$\tilde{h}_m(\Delta) = \dim_K \tilde{H}_m(\Delta, K).$$

The simplicial homology of Δ is a topological invariant of the topological space (polyhedron) $|\Delta|$ defined by Δ. We say that Δ is *acyclic* if all the Betti numbers, $\tilde{h}_m(\Delta)$ vanish and that Δ is a *bouquet* if $\tilde{h}_m(\Delta) = 0$ for $m \neq r(\Delta) - 1$. The *reduced Euler characteristic* of Δ is the alternating sum:

$$\mu(\Delta) = \sum_{m=-1}^{\infty} (-1)^m \tilde{h}_m(\Delta).$$

Unlike the Betti numbers, $\mu(\Delta)$ does not depend on the field K.

We extend the topological concepts defined above to posets by means of the order complex. For example, $\mu(P) = \mu(\Delta(P))$. The reduced Euler characteristic of P may also be computed using the *incidence algebra* $I(\hat{P}, \mathbb{Q})$, as defined by Rota [15], where $\mathbb{Q}$ is the field of rational numbers. In Rota's notation, $\mu(P) = \mu(\hat{0}, \hat{1})$, where μ is the Möbius function of $I(\hat{P}, \mathbb{Q})$.

As one last bit of notation, we write $[n]$ for the set $\{1, 2, \ldots, n\}$.

2. Cohen–Macaulay Connectivity

In this section we define and develop the concept of Cohen–Macaulay (CM) connectivity, using the algebraic topological point of view. Later, in Section 4, we consider an alternative approach utilizing ring theory.

Roughly speaking, a CM complex is a complex that is locally a bouquet. More precisely, for a simplicial complex Δ and a simplex $\sigma \in \Delta$, define

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \varnothing \text{ and } \tau \cup \sigma \in \Delta\};$$

we then say that Δ is *Cohen–Macaulay*, if $\mathrm{link}_\Delta(\sigma)$ is a bouquet for every $\sigma \in \Delta$ (including $\sigma = \varnothing$). A poset P is said to be CM if for every $x \leq y$ in $\hat{P}$, the open interval (x, y) is a bouquet. It is easy to prove that P is CM if and only if $\Delta(P)$ is CM (see [3, Proposition 3.3]).

A simplicial complex Δ is said to be *Almost Cohen–Macaulay* (ACM) if $\mathrm{link}_\Delta(\sigma)$ is a bouquet for every $\sigma \in P(\Delta)$. A poset P is said to be ACM if $\Delta(P)$ is ACM. For example, if $|\Delta|$ is a manifold, then Δ is ACM but would not in general be CM.

Let $k>0$ be an integer, and let Δ be a simplicial complex on vertex set V. We say that Δ is *k-Cohen–Macaulay connected* (or simply k-CM) if for every subset $T \subseteq V$ such that $|T|<k$, we have:

(i) $\Delta\backslash T$ is CM, and

(ii) $r(\Delta\backslash T)=r(\Delta)$.

A poset P is said to be k-CM if $\Delta(P)$ is k-CM. Equivalently, P being k-CM means that $P\backslash S$ is CM of the same rank as P whenever $S \subseteq P$ has fewer than k elements. Note that $\hat{P}$ is CM if and only if P is CM but that $\hat{P}$ is never 2-CM even if P is so.

We now consider what it means for a connected graph Γ to be k-CM. Let $\Delta(\Gamma)$ be the circuit matroid of Γ. Then $\Delta(\Gamma)$ is totally pure and hence CM (see Björner [9]). Now for any subset T of the edge set E of Γ, $\Delta(\Gamma)\backslash T=\Delta(\Gamma|(E\backslash T))$ is also totally pure and hence CM. On the other hand, $r(\Delta(\Gamma)\backslash T)=r(\Delta(\Gamma))$ if and only if T does not disconnect Γ. Therefore, if $k \geq 2$, $\Delta(\Gamma)$ is k-CM if and only if Γ is k-connected.

Since it is the rank condition that is crucial in the case of graphs, it is useful to rephrase this condition in some other ways.

THEOREM 2.1

(a) *Let Δ be a simplicial complex on vertex set V, and let k be a positive integer for which $\Delta\backslash T$ is CM whenever $|T|<k$.*
Then the following are equivalent:

(i) *Δ is k-CM*;

(ii) *for every non-maximal $\sigma \in \Delta$, the number of vertices in $\mathrm{link}_\Delta(\sigma)$ is at least k.*

(b) *Let P be a poset, and let k be a positive integer for which $P\backslash T$ is CM whenever $|T|<k$.*
Then the following are equivalent:

(i) *P is k-CM*;

(ii) *for every $j \in [r(P)]$, $|\{x \in P \mid r(x)=j\}| \geq k$.*

PROOF. We first observe that k-CM is a local property, i.e. if Δ is k-CM then for every $\sigma \in \Delta$, $\mathrm{link}_\Delta(\sigma)$ is also k-CM. This follows immediately from the fact that $\mathrm{link}_\Delta(\sigma)\backslash T=\mathrm{link}_{\Delta\backslash T}(\sigma)$ for any subset T of the vertex set of $\mathrm{link}_\Delta(\sigma)$. Therefore (a(i)) implies (a(ii)).

Suppose that Δ satisfies condition (a(ii)) and that $T \subseteq V$ is a subset having fewer than k elements, which is a minimal counter-example to the rank condition, that is, $r(\Delta\backslash T)=r(\Delta)-1$ but $r(\Delta\backslash T')=r(\Delta)$ for every proper subset T' of T. Since $r(\Delta\backslash T)=r(\Delta)-1$, each maximal simplex $\sigma \in \Delta\backslash T$ has $r(\Delta)-1$ elements. Choose one such simplex and consider $\mathrm{link}_\Delta(\sigma)$. Since $r(\Delta)=|\sigma|+1$, $\mathrm{link}_\Delta(\sigma)$ has rank 1 and by condition (a(ii)) it has at least k vertices. Now let v be a vertex of $\mathrm{link}_\Delta(\sigma)$. If $v \notin T$, then $\sigma \cup \{v\} \in \Delta\backslash T$ which contradicts the maximality of σ. Therefore every vertex of $\mathrm{link}_\Delta(\sigma)$ is in T. This contradicts $|T|<k$. Part (a) then follows.

Now consider part (b). Clearly (b(i)) implies (b(ii)). Suppose that (b(ii)) holds but that (b(i)) does not. Let $S \subseteq P$ be a minimal counter-example to the rank condition. Then we have $r(P\backslash S)=r(P)-1$, but $r(P\backslash S')=r(P)$ for any proper subset S' of S. Since $P\backslash S$ and $P\backslash S'$ are CM, both are ranked. Let $x \in S$ and let $S'=S\backslash\{x\}$. Then every maximal chain of $P\backslash S$ has $r(P)-1$ elements while every maximal chain of $(P\backslash S)\cup\{x\}$ has $r(P)$ elements, and hence contains x. If $y \in (P\backslash S)\cup\{x\}$ were another element having the same rank as x, then no maximal chain through y would contain x. It follows that S contains every element of the same rank as x. This contradicts condition (b(ii)). Thus (b(ii)) implies (b(i)), and we are done.

For the rest of this section we consider some operations on simplicial complexes and show that k-CM connectivity is preserved by these operations.

RANK SELECTION

Let P be a partially ordered set. For $S \subseteq [r(P)]$ we write P_S for $\{x \in P | r(x) \in S\}$ and call it a *rank-selected subposet* of P. This concept generalizes to simplicial complexes as follows. Let Δ be a simplicial complex on the vertex set V. We say that Δ is *balanced of type* (a, b), if a and b are integers such that $a + b = r(\Delta)$ and if V may be partitioned into subsets U, W such that for every $\sigma \in \Delta$ we have $|\sigma \cap U| \leq a$ and $|\sigma \cap W| \leq b$ (see Stanley [18]). In this case we call $\Delta | U$ a *rank-selected subcomplex* of Δ. It is clear that if P is a poset and $S \subseteq [r(P)]$ then $\Delta(P_S)$ is a rank-selected subcomplex of $\Delta(P)$.

A basic result in the theory of CM complexes is that the CM property is preserved by rank-selection. This was shown in [3, Theorem 6.4]. An immediate consequence of this result is that k-CM connectivity is also preserved by rank selection.

THEOREM 2.2. *Let Δ be a k-CM complex on the vertex set V, and let $\Delta | U$ be a rank-selected subcomplex for some $U \subseteq V$. Then $\Delta | U$ is also k-CM.*

Another kind of rank selection is the "d-skeleton" of topology. More precisely, if Δ is a simplicial complex, the d-skeleton is the complex

$$\Delta_{d+1} = \{\sigma \in \Delta \,|\, |\sigma| \leq d + 1\}.$$

It is straightforward to verify that if Δ is k-CM then Δ_{d+1} is also.

THE JOIN

Let Δ and Δ' be simplicial complexes on disjoint vertex sets V and V'. Their *join* is the complex on $V \cup V'$ given by:

$$\Delta * \Delta' = \{\sigma \cup \tau | \sigma \in \Delta \text{ and } \tau \in \Delta'\}.$$

If P and Q are posets, then $\Delta(P) * \Delta(Q) = \Delta(P \oplus Q)$, where $P \oplus Q$ is the *ordinal sum* of P and Q as defined in [8].

THEOREM 2.3. *Let Δ and Δ' be k-CM complexes. Then $\Delta * \Delta'$ is also k-CM.*

PROOF. It is well known that the join of bouquets is a bouquet. See, for example, [7, Proposition 3.2]. The result follows from this and from these two facts:

(i) if $T \subseteq V \cup V'$, then $(\Delta * \Delta') \backslash T = (\Delta \backslash (T \cap V)) * (\Delta \backslash (T \cap V'))$;

(ii) if $\sigma \in \Delta$ and $\tau \in \Delta'$, then $\operatorname{link}_{\Delta * \Delta'}(\sigma \cup \tau) = \operatorname{link}_\Delta(\sigma) * \operatorname{link}_{\Delta'}(\tau)$.

FIBRATION

Let $f: P \to Q$ be an order-preserving map. For some applications it is useful to regard f as a "fibration" that "constructs" the poset P by piecing together the "fibers" $f/y = \{x \in P | f(x) \geq y\}$ for $y \in Q$ (see the discussion in [3, Section 5]). The following result extends [3, Theorem 5.2].

THEOREM 2.4. *Let $f: P \to Q$ be an order-preserving map of posets. Assume that*
(i) $r(P) \geq r(Q)$;
(ii) *Q is CM and either $r(P) = r(Q)$ or Q is acyclic*;
(iii) *for every $y \in Q$, the fiber f/y is k-CM*;
(iv) *for every $y \in Q$, $r(P) - r(f/y) = r(y) - 1$.*
Then P is k-CM.

PROOF. Let $T \subseteq P$ be a subset such that $|T| < k$. Let $y \in Q$ be a minimal element. Then by (iv), $r(P) = r(f/y)$. Now f/y is k-CM so $(f/y)\backslash T$ has rank $r(f/y) = r(P)$. Thus there is a maximal chain of size $r(P)$ in f/y, and hence in P, which misses T. Therefore, $r(P\backslash T) = r(P)$.

It is trivial to verify that the hypotheses of [3, Theorem 5.2] hold for the restriction of f to $P\backslash T$. Therefore $P\backslash T$ is CM. The result now follows.

BARYCENTRIC SUBDIVISION

Let Δ be a finite simplicial complex on vertex set V, and let $P = P(\Delta)$ be the associated poset of non-empty simplices of Δ. The order complex $\Delta(P)$ is the simplicial complex corresponding to the barycentric subdivision of the triangulation of $|\Delta|$ defined by Δ. Now it is relatively easy to show that Δ is CM if and only if $\Delta(P)$ is so (see [3, Proposition 3.3]). The corresponding property for k-CM is false for $k > 2$. Indeed this failure is complete: if $r(\Delta) > 1$ then $\Delta(P(\Delta))$ can be at most 2-CM. To see this, let σ be a simplex of Δ such that $|\sigma| = r$. Let v_1, v_2 be distinct vertices of σ. Let $\Delta' = \Delta(P(\Delta)\backslash\{\{v_1\}, \{v_2\}\})$. Then, if $r > 2$, $\mathrm{link}_{\Delta'}(\sigma) = \Delta((\hat{0}, \sigma)\backslash\{\{v_1\}, \{v_2\}\})$, but $(\hat{0}, \sigma)\backslash\{\{v_1\}, \{v_2\}\}$ is not a ranked poset so $\mathrm{link}_{\Delta'}(\sigma)$ is not pure and hence not CM. If $r = 2$, then $r(\Delta') < r(\Delta)$. In either case, therefore $\Delta(P(\Delta))$ is not 3-CM.

On the other hand, we have a better situation for 2-CM:

THEOREM 2.5. *Let Δ be a simplicial complex on vertex set V. Then Δ is 2-CM if and only if $\Delta(P(\Delta))$ is 2-CM.*

PROOF. We may assume by an obvious induction that the result holds for simplicial complexes having fewer simplices than Δ. Write $P = P(\Delta)$ and $r = r(\Delta) = r(P)$. By [3, Proposition 3.3] we may assume that both Δ and $\Delta(P)$ are CM.

Assume that Δ is 2-CM. Let $\sigma \in P$ and $v \in \sigma$. Define $Q = \{\tau \in P \mid v \in \tau\}$. Then $r(\Delta\backslash\{v\}) = r(P\backslash Q) = r$, while $P\backslash Q \subseteq P\backslash\{\sigma\} \subseteq P$, so that $r(P\backslash\{\sigma\}) = r$ also. Furthermore, $P(\Delta\backslash\{v\}) = P\backslash Q$ is CM. Consider the long exact sequence of the pair $(P\backslash\{\sigma\}, P\backslash Q)$:

(*) $\cdots \to \tilde{H}_j(P\backslash\{\sigma\}) \to \tilde{H}_j(P\backslash Q) \to H_j(P\backslash\{\sigma\}, P\backslash Q) \to \cdots$. Now if $\sigma = \{v\}$, then $H_j(P\backslash\{\sigma\}, P\backslash Q) = 0$ for every j by excision. If $\sigma \neq \{v\}$, then by a more complicated excision we have that

(**) $H_j(P\backslash\{\sigma\}, P\backslash Q) \cong \tilde{H}_{j-1}(P(\mathrm{link}_\Delta(v))\backslash\{\sigma_0\})$, where $\sigma_0 = \sigma\backslash\{v\}$. An alternative way to compute (**) is to use a standard spectral sequence argument. For example, one can apply [2, Corollary 4.2], where P, Q, $P +_A Q$ and D (in the notation of [2]) are taken to be $(P\backslash Q)^* \cup \{\hat{1}\}$, $(Q\backslash\{\sigma\})^*$, $(P\backslash\{\sigma\})^* \cup \{\hat{1}\}$ and the skyscraper diagram $K[\hat{1}]$, respectively. The conclusion of [2, Corollary 4.2] then yields the long exact sequence (*) along with (**).

Now $\mathrm{link}_\Delta(v)$ is 2-CM, so by the inductive hypothesis $P(\mathrm{link}_\Delta(v))$ is also 2-CM. Hence $P(\mathrm{link}_\Delta(v))\backslash\{\sigma_0\}$ is a bouquet of rank $r-1$. By (*), $P\backslash\{\sigma\}$ is also a bouquet. It remains to show that $P\backslash\{\sigma\}$ is ACM, but this follows very easily from the inductive hypothesis. Since σ was arbitrary, it follows that $\Delta(P)$ is 2-CM.

Conversely suppose that $\Delta(P)$ is 2-CM. Let $v \in V$, and set $Q = \{\tau \in P \mid v \in \tau\}$ as above. Apply (*) with $\sigma = \{v\}$. In this special case we have $\tilde{H}_j(P\backslash\{\sigma\}) \cong \tilde{H}_j(P\backslash Q)$ for every j.

Since $\Delta(P)$ is 2-CM, we have that $\Delta(P\backslash\{\sigma\})$ is CM and hence that $P(\Delta\backslash\{v\})=P\backslash Q$ is a bouquet. To show that $\Delta\backslash\{v\}$ is CM it remains to check that $\operatorname{link}_\Delta(\tau)\backslash\{v\}$ is a bouquet for every $\tau\in P$. Now it is easy to check that $\Delta(P(\operatorname{link}_\Delta(\tau)\backslash\{v\}))\cong\Delta((\tau,\hat{1})\backslash\{\tau^0\})$, where $\tau^0=\tau\cup\{v\}$; and we know that $\Delta((\tau,\hat{1})\backslash\{\tau^0\})$ is a bouquet because $\Delta(P)$ is 2-CM. Thus $\Delta\backslash\{v\}$ is CM as desired.

Finally it remains to show that $r(\Delta\backslash\{v\})=r$. Suppose that $r(\Delta\backslash\{v\})\neq r$. Then $r(\Delta\backslash\{v\})=r-1$. Choose a maximal simplex τ of $\Delta\backslash\{v\}$. Since Δ is CM of rank r, τ is not maximal in Δ. Therefore $\rho=\tau\cup\{v\}$ is in Δ and is a maximal simplex. Choose any simplex $\tau_1\subset\cdots\subset\tau_{r-1}=\tau$ in $\Delta(P)$. Now $\Delta(P)\backslash\{\rho\}$ is CM of rank r. Therefore we can find a maximal simplex of $\Delta(P)\backslash\{\rho\}$ containing $\tau_1\subset\cdots\subset\tau_{r-1}$. Such a maximal simplex must have the form $\tau_1\subset\cdots\subset\tau_{r-1}\subset\tau_r$, where τ_r is a maximal simplex of Δ and $\tau_r\neq\rho=\tau\cup\{v\}$. Therefore $v\notin\tau_r$ and hence $\tau_r\in\Delta\backslash\{v\}$. This contradicts the fact that $r(\Delta\backslash\{v\})=r-1$, and completes the proof.

In an early version of this paper, it was conjectured that the 2-CM property is a topological invariant of $|\Delta|$. This has now been shown by Walker [20].

3. Geometric Lattices

In this section we give a simple characterization of k-CM connectivity for semimodular lattices. In particular we find that the doubly CM connected, semimodular lattices are precisely the geometric lattices. We assume that the reader is familiar with the elementary theory of matroids (combinatorial geometries) as presented in Crapo–Rota [12].

Theorem 3.1. *Let $L=\hat{P}$ be a semimodular lattice. The following are equivalent.*
(i) *L is geometric;*
(ii) *P is 2-CM.*

Proof. In a geometric lattice L every non-empty open interval has at least two elements. That (i) implies (ii) then follows from Theorem 2.1 and [3, Corollary 4.3]. Conversely, suppose that P is 2-CM but that L is not geometric. Then L has a join-irreducible x which is not an atom. Let y be the unique element covered by x. Since x is not an atom, y is in P. Since $r(P\backslash\{y\})=r(P)$ by condition (ii), $P\backslash\{y\}$ contains a maximal chain of length strictly less than $r(P\backslash\{y\})$. Therefore $P\backslash\{y\}$ is not a ranked poset and hence not CM. This contradicts condition (ii). The result then follows.

More generally, for $\hat{P}$ a semimodular lattice if P is k-CM, then every non-empty open interval of $\hat{P}$ has at least k elements. We will show the converse. In the discussion to follow we will use the following notation. If $\hat{P}$ is a geometric lattice and $x\in\hat{P}$, we write $A(x)$ for the set of atoms of $\hat{P}$ lying below x. More generally, if $S\subseteq\hat{P}$, we write $A(S)$ for $\bigcup_{x\in S}A(x)$. In matroid theory, x and $A(x)$ are identified, and a subset of the form $A(x)$ is called a *flat* of the matroid. We begin with a technical lemma.

Lemma 3.2. *Let $\hat{P}$ be a geometric lattice. Let $S\subseteq P$ be an antichain, and let $x\in P$ be an element which covers some element of S. If $A(x)\subseteq A(S)$, then $\hat{P}$ has a line containing at most $|S|$ atoms.*

Proof. Let $y_1\in S$ be an element covered by x. Label the rest of the elements of S by y_i, $2\leqslant i\leqslant s=|S|$. Suppose that for some $i\geqslant 2$, $A(x)\subseteq A(S\backslash\{y_i\})$. Then we could replace S by $S\backslash\{y_i\}$ and obtain an even better result. Thus we may assume that $A(x)\nsubseteq A(S\backslash\{y_i\})$ for any $i\geqslant 2$. We remark that $A(x)\subseteq A(S)$ implies that S has at least two elements, since x covers y_1.

We wish to choose two elements $z_1, z_2 \in S$ such that

$$A(x) \not\subseteq A(S \backslash \{z_i\}), \quad \text{for } i = 1, 2. \tag{$*$}$$

If $|S| > 2$, this is trivial. If $|S| = 2$, then we must take $z_i = y_i$. The condition $(*)$ for this case is simply that $A(x) \not\subseteq A(y_i)$, for $i = 1, 2$. Now $A(x) \not\subseteq A(y_1)$ because x covers y_1. Suppose that $A(x) \subseteq A(y_2)$. Then $x \leqslant y_2$, but $x \geqslant y_1$ so this would contradict the assumption that S is an antichain. Thus we may choose z_1 and z_2 in all cases.

Now choose atoms a_i such that $a_i \in A(x) \backslash A(S \backslash \{z_i\})$. Since $A(x) \subseteq A(S)$, it follows that $a_i \in A(z_i)$. Let $b = a_1 \vee a_2$. Then $b \leqslant x$ but b is not below any element of S, for if it were then both a_1 and a_2 would be below that element. Now each atom $a \in A(b)$ is necessarily in $A(x)$ and hence in $A(S)$. Therefore $a \in A(y)$ for some $y \in S$. We claim that no two atoms in $A(b)$ can be in the same set $A(y)$ for any $y \in S$, for if this were so then their supremum b would also be in $A(y)$, and we already noted that this is not possible. Therefore the number of atoms in $A(b)$ cannot exceed $|S|$.

For a finite poset P, its *Dilworth number* $\delta(P)$ is the size of the largest antichain in P or equivalently, by Dilworth's Theorem, the smallest number l such that P is the union of l chains. In the next result we make use of the concept of lexicographic shellability. For the definition and basic properties see Björner [9].

THEOREM 3.3. *Let $\hat{P}$ be a geometric lattice and k a positive integer. The following are equivalent*:

(i) *P is k-CM*;
(ii) *every line of P has at least k points*;
(iii) *if $S \subseteq P$ satisfies $\delta(S) < k$, then $r(P \backslash S) = r(P)$ and $P \backslash S$ is shellable.*

We will make use of a result of Björner [9, Proposition 2.8] which for geometric lattices may be stated as follows. Let $\hat{P}$ be a finite geometric lattice. Choose a total order "$\preccurlyeq$" on the set $A(\hat{1})$ of atoms of $\hat{P}$. Let Q be a subset of P. A triple of elements $x < y < z$ of $\hat{P}$ such that $x < y < z$ is a maximal chain of $[x, z]$ is called a *link* of $\hat{P}$. We define the *label* of a link $x < y < z$ to be the pair of atoms (a, b) such that a is the earliest atom in $A(y) \backslash A(x)$ and b is the earliest atom in $A(z) \backslash A(y)$.

PROPOSITION 3.4 (Björner shellability criterion). *Let $\hat{P}$ be a geometric lattice and $Q \subseteq P$. If there is a total order "$\preccurlyeq$" on $A(\hat{1})$ which satisfies*:

if a link $x < y < z$ of $\hat{P}$ with label (a, b) satisfies
$x \in \hat{Q}$ and $a \prec b$, then it also satisfies $y \in Q$,

then Q is shellable and has the same rank as P.

PROOF OF THEOREM 3.3. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are trivial. Suppose that P satisfies condition (ii) and that $S \subseteq P$ satisfies $\delta(S) < k$. Partition S into antichains as follows. Let T_1 be the set of maximal elements of S. More generally, if $T_1, \ldots, T_n$ are defined, we define T_{n+1} to be the set of maximal elements of $S \backslash \bigcup_{i=1}^{n} T_i$. The condition $\delta(S) < k$ implies that $|T_n| < k$ for all n. By Lemma 3.2 and condition (ii), every $x \in \hat{P}$ which covers an element of T_n satisfies $A(x) \not\subseteq A(T_n) = A(T_n \cup T_{n+1} \cup \cdots)$. Let A_n be $A(\hat{1}) \backslash A(T_n)$. Then $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of sets of atoms which eventually is equal to $A(\hat{1})$ since $A(T_n)$ is empty for n sufficiently large. We total order $A(\hat{1})$ so that every element of A_n precedes every element of $A(\hat{1}) \backslash A_n = A(T_n)$ for all n. We show that this order satisfies Björner's criterion for $Q = P \backslash S$. Let $x < y < z$ be a link of $\hat{P}$, with label (a, b) such that $x \in \hat{Q}$ and $a \prec b$. Suppose that $y \notin Q$, i.e. that $y \in S$. Choose

n so that $y \in T_n$. Now z covers y so $A(z) \not\subseteq A(T_n)$ or equivalently $A(z) \cap A_n \neq \emptyset$. By the choice of the total order on $A(\hat{1})$, every element of A_n precedes every element of $A(T_n)$. Since b is the earliest atom in $A(z) \backslash A(y)$, we have $b \in A_n$. Moreover, $a \in A(y) \subseteq A(T_n)$. Thus b precedes a. This contradicts our assumption that $a < b$. Therefore $y \in Q$, Björner's criterion holds, and $Q = P \backslash S$ is shellable and has the same rank as P.

4. The Stanley–Reisner Ring

In this section we study the Stanley–Reisner ring. The main result is the characterization of k-CM connectivity in terms of a ring-theoretical condition on this ring. We also give some applications of our main result. For other applications see [4, 6, 10].

Let Δ be a simplicial complex on the vertex set V. For each element $a \in V$, we introduce an indeterminate X_a, and we write $K[X_a | a \in V]$ (or simply $K[X]$) for the (free) polynomial ring on these indeterminates. For any subset $S \subseteq V$ we write X_S for the monomial $X_S = \prod_{a \in S} X_a$ in $K[X]$. Let $I_\Delta \subseteq K[X]$ be the ideal generated by $\{X_S | S \subseteq V$ is *not* a simplex of $\Delta\}$. The *Stanley–Reisner* ring of Δ is the quotient ring $K[\Delta] = K[X]/I_\Delta$. This ring was introduced independently by Stanley [16] and by Reisner [14]. Stanley introduced it as a means of proving the Upper Bound Theorem for simplicial spheres. Reisner studied this ring in his thesis at the suggestion of his advisor, Hochster. We will write $K[P]$ for $K[\Delta(P)]$.

In general, let R be a polynomial ring over K, which is represented as a quotient $K[X]/I$ of a free polynomial ring $K[X] = K[X_i | 1 \leq i \leq n]$. The direct sum of m copies of $K[X]$ will be denoted $K[X]^m$. A *finite free resolution* of R over $K[X]$ is an exact sequence of the form:

$$0 \to K[X]^{m_l} \xrightarrow[\phi_l]{} K[X]^{m_{l-1}} \xrightarrow[\phi_{l-1}]{} \cdots \xrightarrow[\phi_2]{} K[X]^{m_1} \xrightarrow[\phi_1]{} K[X] \xrightarrow[\phi_0]{} R \to 0,$$

where each ϕ_i, $1 \leq i \leq l$, is a $K[X]$-homomorphism and ϕ_0 is the natural homomorphism $\phi_0 : K[X] \to K[X]/I$. One can show that there is a finite free resolution which simultaneously minimizes all the numbers m_i and that the resulting resolution is essentially unique. The numbers so obtained are called the *Betti numbers* of R over $K[X]$, and they will be denoted simply b_i. In homological algebra the b_i are given by

$$b_i = \dim_K \operatorname{Tor}_i^{K[X]}(R, K).$$

Note that b_0 is always 1.

The Hilbert Syzygy Theorem states that $b_i = 0$ for $i > n$, where n is the number of indeterminates occurring in $K[X]$, i.e. the length of the minimal free resolution of R is at most n. The number n is called the *Krull dimension* of $K[X]$. This concept can be generalized to arbitrary rings. Let Spec(R) denote the partially ordered set of prime ideals of R. The *Krull dimension* of R is one less than the rank of Spec(R) as a poset. One can show that in general if l is the Krull dimension of R then $b_i \neq 0$ for $0 \leq i \leq n - l$. In other words the length of the minimal free resolution of R over $K[X]$ is at least $n - l$. If the length of the minimal free resolution is precisely $n - l$, i.e. if $b_i = 0$ for $i > n - l$, we say that R is *Cohen–Macaulay*. The extent to which R is close to being Cohen–Macaulay is measured by the *depth* of R: if i is the largest integer for which $b_i \neq 0$, then depth$(R) = n - i$. Thus in general we have $0 \leq \text{depth}(R) \leq \text{Krull dim}(R)$, and R is CM if and only if depth$(R) = $ Krull dim(R). When R is CM, we call b_{n-l} the *type* of R. We say R is *Gorenstein* if R is CM and has type 1.

The Krull dimension of the Stanley–Reisner ring $K[\Delta]$ is easily seen to be $r(\Delta)$. Thus $K[\Delta]$ is CM if and only if $b_{N-r(\Delta)+1} = 0$, where $N = |V|$ is the number of vertices of Δ. Reisner's Theorem links the concept of a CM complex with the ring-theoretic concept:

THEOREM 4.1 (Reisner [14]). *Let Δ be a simplicial complex. Then $K[\Delta]$ is CM if and only if Δ is CM.*

Shortly after Reisner proved this theorem, Hochster went on to give a formula for every Betti number of the Stanley–Reisner ring, which provides another topological characterization of CM complexes.

THEOREM 4.2 (Hochster [13]). *Let Δ be a simplicial complex on the vertex set V. The kth Betti number of $K[\Delta]$ is given by*

$$b_k = \sum_{U \subseteq V} \tilde{h}_{|U|-1-k}(\Delta|U).$$

In the special case of a simplicial complex of the form $\Delta = \Delta(P)$, for a poset P, there is yet another topological characterization of the CM property.

THEOREM 4.3 (Baclawski–Garsia [7, Corollary 5.2]). *Let P be a poset. Then $K[P]$ is CM if and only if for every $S \subseteq [r(P)]$,*

$$\tilde{h}_{|S|-1}(P_S) = (-1)^{|S|-1}\mu(P_S).$$

For reference, we collect together the various criteria for a poset to be CM in the following corollary.

COROLLARY 4.4 (Baclawski, Garsia, Hochster, Reisner). *For a poset P, the following are equivalent:*

(i) *P is CM;*

(ii) *for every pair $x \leqslant y$ in $\hat{P}$ and every $m < r(x, y) - 1$, $\tilde{h}_m((x, y)) = 0$;*

(iii) *for every subset $T \subseteq P$ and every $m < r(P) - |T| - 1$, $\tilde{h}_m(P\backslash T) = 0$;*

(iv) *for every subset of $T \subseteq P$, $\tilde{h}_{r(P)-|T|-2}(P\backslash T) = 0$;*

(v) *for every pair $x \leqslant y$ in $\hat{P}$, for every $T \subseteq (x, y)$ and for every $m < r(x, y) - |T| - 1$, $\tilde{h}_m((x, y)\backslash T) = 0$;*

(vi) *for every subset $S \subseteq [r(P)]$, $\tilde{h}_{|S|-1}(P_S) = (-1)^{|S|-1}\mu(P_S)$.*

PROOF. The equivalence (i) $\Leftrightarrow$ (ii) follows by definition, while (i) $\Leftrightarrow$ (iv) follows from Theorems 4.1 and 4.2. These also give (i) $\Leftrightarrow$ (iii). Combining (ii) and (iii) gives (i) $\Leftrightarrow$ (v). Finally (i) $\Leftrightarrow$ (vi) follows from Theorem 4.3.

We now come to our main result.

THEOREM 4.5. *For a poset P and positive integer k, the following are equivalent:*

(i) *P is k-CM;*

(ii) *for every $T \subseteq P$ and $m \leqslant \min(r(P) - 2, r(P) - 3 + k - |T|)$, $\tilde{h}_m(P\backslash T) = 0$ and if $|T| < k$, then $r(P\backslash T) = r(P)$;*

(iii) *for all $j < k - 1$,*

$$b_{N-r-j}(K[P]) = (-1)^r \sum_{n=1}^{j+1} (-1)^n \binom{N+2}{j-n+1} \mu^n(\hat{0}, \hat{1}),$$

where $r = r(P)$, $N = |P|$, μ is the Möbius function in the incidence algebra $I(\hat{P}, \mathbb{Q})$, and $\mu^n = \mu * \cdots * \mu$ is the n-th convolution of μ with itself in $I(\hat{P}, \mathbb{Q})$.

The expression in (iii) involving μ may also be written as

$$(-1)^{r-1}\sum_{m=0}^{j}(-1)^m\binom{N-m}{j-m}\hat{\mu}^{m+1}(\hat{0},\hat{1}),$$

where $\hat{\mu}=\mu-\delta\in I(\hat{P},\mathbb{Q})$. Several other formulas equivalent to this one are found in the course of the proof.

PROOF. We begin by applying Corollary 4.4 to the definition of k-CM connectivity.

(i) $\Leftrightarrow$ (b) $\forall S\subseteq P$ s.t. $|S|<k$, $(r(P\backslash S)=r$ and $P\backslash S$ is CM),

$\Leftrightarrow$ (c) $\forall S\subseteq P$ s.t. $|S|<k$, $(r(P\backslash S)=r$ and $\forall T\subseteq P\backslash S, \forall m\leqslant r-2-|T|$ $(\tilde{h}_m(P\backslash(S\cup T))=0))$.

We next change the quantifiers over S and T to ones over $U=S\cup T$ and S:

(c) $\Leftrightarrow$ (d) $\forall U\subseteq P, \forall S\subseteq U$ s.t. $|S|<k$, $(r(P\backslash S)=r$ and $\forall m\leqslant r-2-|U|+|S|$ $(\tilde{h}_m(P\backslash U)=0))$.

The condition $\tilde{h}_m(P\backslash U)=0$ does not depend on S while the range of m is determined only by $|S|$. If $|U|\geqslant k-1$, then there exists a $(k-1)$-element subset of U, so the constraint on m is that $m\leqslant r-3-|U|+k$. If $|U|<k-1$, then the largest $S\subseteq U$ we could choose has cardinality $|U|$. The constraint on m then becomes simply $m\leqslant r-2$. Therefore the constraint on m in general is that $m\leqslant\min(r-2, r-3+k-|U|)$, the first term being smaller when $|U|<k-1$ and the second being so when $|U|\geqslant k-1$. Thus

(d) $\Leftrightarrow$ (e) $\forall U\subseteq P, \forall m\leqslant\min(r-2, r-3+k-|U|)$, $(\tilde{h}_m(P\backslash U)=0)$
and $\forall S\subseteq P$ s.t. $|S|<k$, $(r(P\backslash S)=r)$.

This is clearly equivalent to condition (ii).

We now consider condition (iii). Theorem 4.2 tells us that the ith Betti number of $K[P]$ over $K[X]$ is given by

$$b_i=\sum_{T\subseteq P}\tilde{h}_{N-1-i-|T|}(P\backslash T).$$

If we set i equal to $N-r-j$, we get

$$\text{(f)}\qquad b_{N-r-j}=\sum_{\substack{T\subseteq P\\ |T|\geqslant j}}\tilde{h}_{r-1-|T|+j}(P\backslash T).$$

The constraint on T stems from the fact that $r(P\backslash T)\leqslant r$ so that $\tilde{h}_m(P\backslash T)=0$ trivially holds for $m\geqslant r$.

Assume that P satisfies condition (ii). Then the terms on the right-hand side of (f) vanish whenever $r-1-|T|+j\leqslant\min(r-2, r-3+k-|T|)$. This condition is easily seen to be equivalent to the pair of conditions $|T|\geqslant j+1$ and $j<k-1$. Therefore,

$$\text{(ii)}\Rightarrow\text{(g)}\qquad b_{N-r-j}=\sum_{\substack{T\subseteq P\\ |T|=j}}\tilde{h}_{r-1}(P\backslash T),\quad\text{for } j<k-1.$$

Now $\tilde{h}_m(P\backslash T)=0$ when $m<r-1$ and $|T|=j<k-1$ because (ii) $\Rightarrow$ (i) $\Rightarrow$ (b). Therefore $\tilde{h}_{r-1}(P\backslash T)=(-1)^{r-1}\mu(P\backslash T)$, and we have:

$$\text{(g)}\Rightarrow\text{(h)}\qquad b_{N-r-j}=\sum_{\substack{T\subseteq P\\ |T|=j}}(-1)^{r-1}\mu(P\backslash T),\quad\text{for } j<k-1.$$

LEMMA 4.6. *Let T be a subset of a poset P. Then*

$$\mu(P\backslash T)=\sum_{(x_1<x_2<\cdots<x_m)\subseteq T}(-1)^m\mu(\hat{0},x_1)\mu(x_1,x_2)\cdots\mu(x_m,\hat{1}).$$

PROOF. It is easy to verify the case $|T|=1$ either by using the definition of μ or by using Philip Hall's Theorem [15, Corollary 2 of Theorem 3]. This case says simply that $\mu(P\backslash\{x\})=\mu(P)-\mu(\hat{0},x)\mu(x,\hat{1})$. The general case follows from this case by an easy induction.

This lemma implies that

(h)$\Leftrightarrow$(j) $\forall j<k-1$, b_{N-r-j}

$$=\sum_{\substack{T\subseteq P\\ |T|=j}}(-1)^{r-1}\sum_{(x_1<\cdots<x_m)\subseteq T}(-1)^m\mu(\hat{0},x_1)\mu(x_1,x_2)\cdots\mu(x_m,\hat{1})$$

$$=(-1)^{r-1}\sum_{(x_1<\cdots<x_m)\subseteq P}(-1)^m|\{T|(x_1<\cdots<x_m)\subseteq T\subseteq P\text{ and }|T|=j\}|\;\mu(\hat{0},x_1)\cdots\mu(x_m,\hat{1})$$

$$=(-1)^{r-1}\sum_{(x_1<\cdots<x_m)}(-1)^m\binom{N-m}{j-m}\mu(\hat{0},x_1)\mu(x_1,x_2)\cdots\mu(x_m,\hat{1})$$

$$=(-1)^{r-1}\sum_{m\geq 0}(-1)^m\binom{N-m}{j-m}\sum_{(x_1<\cdots<x_m)}\mu(\hat{0},x_1)\mu(x_1,x_2)\cdots\mu(x_m,\hat{1})$$

$$=(-1)^{r-1}\sum_{m\geq 0}(-1)^m\binom{N-m}{j-m}\hat{\mu}^{m+1}(\hat{0},\hat{1}).$$

Now we always have that $\mu^0(\hat{0},\hat{1})=\delta(\hat{0},\hat{1})=0$, so

$$\hat{\mu}^{m+1}(\hat{0},\hat{1})=(\mu-\delta)^{m+1}(\hat{0},\hat{1})=\sum_{n=1}^{m+1}(-1)^{m+1-n}\binom{m+1}{n}\mu^n(\hat{0},\hat{1}).$$

Therefore,

(j)$\Leftrightarrow$(k) $\forall j<k-1$, b_{N-r-j}

$$=(-1)^{r-1}\sum_{m\geq 0}(-1)^m\binom{N-m}{j-m}\sum_{n=1}^{m+1}(-1)^{m+1-n}\binom{m+1}{n}\mu^n(\hat{0},\hat{1})$$

$$=(-1)^r\sum_{m\geq 0}\sum_{n=1}^{m+1}(-1)^n\binom{N-m}{j-m}\binom{m+1}{n}\mu^n(\hat{0},\hat{1})$$

$$=(-1)^r\sum_{n\geq 1}(-1)^n\left(\sum_{m\geq n-1}\binom{N-m}{j-m}\binom{m+1}{n}\right)\mu^n(\hat{0},\hat{1})$$

$$=(-1)^r\sum_{n=1}^{j+1}(-1)^n\binom{N+2}{j-n+1}\mu^n(\hat{0},\hat{1}).$$

The last equality follows from the Chu–Vandermonde convolution (cf. Comtet [11, p. 44]). Therefore (ii)$\Rightarrow$(iii).

We now consider the converse (iii)$\Rightarrow$(ii). Assume that P satisfies (iii). We will use induction on k. In the case $k=1$, condition (iii) implies only that $b_{N-r+1}=0$. By Theorem 4.1 this implies that P is CM, and this case follows. Next suppose that (iii)$\Leftrightarrow$(ii) for

$k'<k$ and that $k>1$. Then because (ii) $\Leftrightarrow$ (i), $P\backslash T$ is CM for $|T|<k-1$. In particular $P\backslash T$ is CM for all T occurring in the sums in (g) and (h). Therefore (g) is equivalent to (h). Since (f) holds in general, it follows that all terms in (f) not occurring in (g) must vanish. It is easy to check that this implies the first part of condition (ii).

We now show that the rank condition holds for P. Let $S\subseteq P$ be a minimal counterexample to the rank condition, so that $r(P\backslash S)=r-1$ and $|S|<k$. Then the Krull dimension of $K[P\backslash S]$ is $r-1$ and $b_{|P\backslash S|-(r-1)}(K[P\backslash S])\neq 0$. Now applying Theorem 4.2 we find that

$$\begin{aligned} b_{|P\backslash S|-(r-1)}(K[P\backslash S]) &= \sum_{T\subseteq P\backslash S} \tilde{h}_{r(P\backslash S)-1-|T|}(P\backslash(S\cup T)) \\ &= \sum_{S\subseteq U\subseteq P} \tilde{h}_{r-2-|U|+|S|}(P\backslash U). \end{aligned}$$

By the first part of condition (ii), these terms vanish when $r-2-|U|+|S|\leqslant \min(r-2, r-3+k-|U|)$, i.e. when $-|U|+|S|\leqslant \min(0, k-1-|U|)$. Now $U\supseteq S$ so $-|U|+|S|\leqslant 0$ always holds. On the other hand, $|S|<k$ so $-|U|+|S|\leqslant k-1-|U|$ also always holds. Thus we have a contradiction, and we conclude that $r(P\backslash S)=r$ for all S such that $|S|<k$. Condition (ii) therefore follows.

Perhaps the most interesting special case of Theorem 4.5 (after Reisner's Theorem, of course, which is the case $k=1$) is the following corollary.

COROLLARY 4.7. *For a poset P, the following are equivalent*:
(i) *P is 2-CM*;
(ii) *$K[P]$ is CM and the type of $K[P]$ is $|\mu(P)|$.*

In general, if P is CM, then the type of $K[P]$ is at least $|\mu(P)|$. Thus one may regard a 2-CM poset as being a CM poset for which $K[P]$ has "minimum type", or equivalently in Stanley's terminology [17, p. 54] a CM poset for which $K[P]$ is a "level ring". In particular a non-acyclic CM poset is Gorenstein if and only if it is 2-CM and satisfies $\mu(P)=\pm 1$.

If $\hat{P}$ is a geometric lattice, then by Theorem 3.1, P is 2-CM and hence the type of $K[P]$ is $|\mu(P)|$. More generally, by Theorem 3.3, the smallest line of $\hat{P}$ has k points if and only if the Betti numbers of $K[P]$ satisfy:

$$\begin{aligned} b_{M-R-1} &= |\mu(P)| \\ &\vdots \\ b_{M-R+l} &= \left|\sum_{n=1}^{l} (-1)^n \binom{M}{l-n} \mu^n(\hat{0},\hat{1})\right| \\ &\vdots \\ b_{M-R-k-1} &= \left|\sum_{n=1}^{k-1} (-1)^n \binom{M}{k-n-1} \mu^n(\hat{0},\hat{1})\right| \\ b_{M-R-k} &> \left|\sum_{n=1}^{k} (-1)^n \binom{M}{k-n} \mu^n(\hat{0},\hat{1})\right|, \end{aligned}$$

where M is the number of flats (including improper flats) of $\hat{P}$, and R is the rank of $\hat{P}$ as a combinatorial geometry, i.e. $R=r(P)+1$.

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